Accuracy of Microwave Cavity Perturbation Measurements

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***Abstract*—Techniques based on the perturbation of cavity resonators are commonly used to measure the permittivity and permeability of samples of dielectric and ferrite materials at microwave frequencies. They are also used to measure the local electric- and magnetic-field strengths in microwave structures including the shunt impedances of cavity resonators and the coupling impedances of slow-wave structures. This paper reexamines the assumptions made in the theory of these techniques and provides estimates of the errors of measurement arising from them.**

***Index Terms*—Cavity perturbation measurements, microwave measurements.**

I. Introduction

When a small object is introduced into a microwave cavity resonator, the resonant frequency is perturbed [1], [2]. Since it is possible to measure the change in frequency with high accuracy, this provides a valuable method for measuring the electric and magnetic properties of the object if the properties of the cavity are known, or for characterizing the cavity if the properties of the perturber are known. Techniques based upon this principle are in common use for measuring the dielectric and magnetic properties of materials at microwave frequencies [3]. They also used for measuring the local electric- and magnetic-field strengths within microwave structures and especially for finding the shunt impedances of cavity resonators for use in klystrons and particle accelerators and the coupling impedances of slow-wave structures for use in traveling-wave tubes and linear accelerators [4]–[6]. The theoretical basis of these measurements is well established, but involves some simplifications. This paper reexamines these assumptions and approximations to show the effect that they have on the accuracy of the measurements.

II. Theory

The theory of the perturbation of cavity resonators has been given by a number of authors. The treatment given here is essentially that presented by Waldron [1], but with some differences that maintain the symmetry of the equations. We shall study the properties of two identical cavity resonators containing nonconducting perturbing objects. Let the fields in the two cavities be $\boldsymbol{E}_0 \exp j\omega_0 t$ and $\boldsymbol{H}_0 \exp j\omega_0 t$ and $\boldsymbol{E}_1 \exp j\omega_1 t$ and $\boldsymbol{H}_1 \exp j\omega_1 t$. Making use of Maxwell's curl equations, we obtain

$$\nabla \times E_0 = -j\omega_0 B_0 \tag{1}$$

$$\nabla \times H_1 = j\omega_1 D_1. \tag{2}$$

Taking the scalar product of $\boldsymbol{H}_1$ with (1) and $\boldsymbol{E}_0$ with (2) and subtracting gives

$$H_1 \cdot (\nabla \times E_0) - E_0 \cdot (\nabla \times H_1) = -j\omega_0 H_1 \cdot B_0 - j\omega_1 E_0 \cdot D_1. \tag{3}$$

However,

$$\nabla \cdot (E_0 \times H_1) = H_1 \cdot (\nabla \times E_0) - E_0 \cdot (\nabla \times H_1). \tag{4}$$

Therefore,

$$\nabla \cdot (E_0 \times H_1) = -j\omega_0 H_1 \cdot B_0 - j\omega_1 E_0 \cdot D_1. \tag{5}$$

Integrating (5) over the volume of the cavity, and making use of Gauss' theorem

$$\iiint \nabla \cdot A\, dv = \iint A \cdot dS \tag{6}$$

yields

$$\iint_S (E_0 \times H_1) \cdot dS = \iiint_V (-j\omega_0 H_1 \cdot B_0 - j\omega_1 E_0 \cdot D_1)\, dv. \tag{7}$$

where S is the surface of the cavity and V its volume. By a similar argument, exchanging the subscripts, we obtain

$$\iint_S (E_1 \times H_0) \cdot dS = \iiint_V (-j\omega_1 H_0 \cdot B_1 - j\omega_0 E_1 \cdot D_0)\, dv. \tag{8}$$

If the walls of the cavity can be regarded as perfectly conducting, then $\boldsymbol{E}$ is normal to the wall and $\boldsymbol{H}$ is tangential to the wall. Thus, the vector products are tangential to the wall and the left-hand sides of (7) and (8) are zero. Equating the right-hand sides of (7) and (8) and rearranging gives

$$j\omega_0 \iiint_V (E_1 \cdot D_0 - H_1 \cdot B_0)\, dv = j\omega_1 \iiint_V (E_0 \cdot D_1 - H_0 \cdot B_1)\, dv. \tag{9}$$

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If we now assume that the cavity with subscript zero is empty and let $\omega_1 = \omega_0 + \triangle\omega$, (9) can be rearranged to give

$$\frac{\Delta\omega}{\omega_0} = \frac{\iiint_V \Big[(E_1 \cdot D_0 - E_0 \cdot D_1) - (H_1 \cdot B_0 - H_0 \cdot B_1)\Big]\, dv}{\iiint_V (E_0 \cdot D_1 - H_0 \cdot B_1)\, dv}. \tag{10}$$

The integrand in the numerator of this equation is zero everywhere outside the volume of the perturbing object. We may, therefore, restrict the volume of integration to the volume of the object denoted by V_1. Thus,

$$\frac{\Delta\omega}{\omega_0} = \frac{\iiint_{V_1} \Big[(E_1 \cdot D_0 - E_0 \cdot D_1) - (H_1 \cdot B_0 - H_0 \cdot B_1)\Big]\, dv}{\iiint_V (E_0 \cdot D_1 - H_0 \cdot B_1)\, dv}. \tag{11}$$

The only assumption that has been made thus far is that the cavity walls are perfectly conducting. There is no restriction on the size or shape of the perturbing object, or of its material, provided that it is not conducting. The symmetry of (11) ensures that its validity is independent of the magnitudes of the fields in the two cavities. For a nonmagnetic object, the second bracket in the numerator of (11) is zero and

$$\frac{\Delta\omega}{\omega_0} = \frac{\iiint_{V_1} (E_1 \cdot D_0 - E_0 \cdot D_1)\, dv}{\iiint_V (E_0 \cdot D_1 - H_0 \cdot B_1)\, dv}. \tag{12}$$

If we set $\boldsymbol{E}_1 = \boldsymbol{E}_0 + \boldsymbol{e}$ and, similarly for the other variables, (11) becomes

$$\frac{\Delta\omega}{\omega_0} = \frac{\iiint_{V_1} \Big[(e \cdot D_0 - E_0 \cdot d) - (h \cdot B_0 - H_0 \cdot b)\Big]\, dv}{\iiint_V (E_0 \cdot D_0 - H_0 \cdot B_0) + (E_0 \cdot d - H_0 \cdot b)\, dv}. \tag{13}$$

Equations (11)–(13) cannot be applied directly because it is not normally possible to find closed-form expressions for the fields in the perturbed cavity. In order to derive a useful formula, certain approximations must be made.

Assumption 1: The perturber is made of homogeneous isotropic material so that $\boldsymbol{D}$ and $\boldsymbol{B}$ can be expressed in terms of $\boldsymbol{E}$, $\boldsymbol{H}$ and the permittivity and permeability of the material. Equation (11) becomes

$$\frac{\Delta\omega}{\omega_0} = \frac{\iiint_{V_1} \Big[\varepsilon_0(1-\varepsilon_r)E_0 \cdot E_1 - \mu_0(1-\mu_r)H_0 \cdot H_1\Big]\, dv}{\iiint_V (E_0 \cdot D_1 - H_0 \cdot B_1)\, dv}. \tag{14}$$

Assumption 2: The perturbation is small so that the second term in the denominator of (13) can be neglected. Equation (14) becomes

$$\frac{\Delta\omega}{\omega_0} = \frac{\iiint_{V_1} \Big[\varepsilon_0(1-\varepsilon_r)E_0 \cdot E_1 - \mu_0(1-\mu_r)H_0 \cdot H_1\Big]\, dv}{\iiint_V (E_0 \cdot D_0 - H_0 \cdot B_0)\, dv}. \tag{15}$$

This assumption has removed the symmetry of the equation so that the frequency perturbation is dependent on the relative amplitudes of the fields in the empty and perturbed cavities. The denominator is recognized as $4W_0$, where W_0 is the stored energy in the empty cavity.

Assumption 3: The perturber is small enough for $\boldsymbol{E}$ and $\boldsymbol{H}$ to be effectively constant within it so that the numerator is equal to the integrand multiplied by the volume of the perturber. Equation (15) becomes

$$\frac{\Delta\omega}{\omega_0} = \frac{\Big[\varepsilon_0(1-\varepsilon_r)E_0 \cdot E_1 - \mu_0(1-\mu_r)H_0 \cdot H_1\Big]V_1}{4W_0}. \tag{16}$$

Assumption 4: The $\boldsymbol{E}$- and $\boldsymbol{H}$-fields outside the perturber are unchanged by its presence and those within the perturber can be determined from the boundary conditions at its surface. This enables simple expressions for the frequency perturbation to be derived in two cases.

A. Long Thin Cylindrical Dielectric Rod Aligned Parallel to $\boldsymbol{E}_0$

Since the tangential electric field is continuous at the surface of the rod ($r = b$), it follows that $\boldsymbol{E}_1 = \boldsymbol{E}_0$, and since $\mu_r = 1$, (16) reduces to the usual approximate formula for perturbation of the frequency by a thin dielectric rod

$$\frac{\Delta\omega}{\omega_0} = \frac{\varepsilon_0(1-\varepsilon_r)|E_{00}|^2\pi b^2 L}{4W_0} \tag{17}$$

where E_{00} is the magnitude of $\boldsymbol{E}_0$ on the axis and L is the length of the rod.

B. Dielectric Sphere

Under the quasi-static approximation, the electric field within a dielectric sphere placed in a uniform external electric field $\boldsymbol{E}_0$ is given by [7]

$$E_1 = \frac{3E_0}{\varepsilon_r + 2}. \tag{18}$$

Substitution of this expression into (16) and taking $\mu_r = 1$ yields the usual expression for the perturbation of the frequency by a small dielectric sphere

$$\frac{\Delta\omega}{\omega} = -\frac{\pi R^3 \varepsilon_0 |E_0|^2}{W_0} \cdot \left\{\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right\} \tag{19}$$

where R is the radius of the sphere.

It is generally assumed that these approximate expressions are accurate enough for most purposes, but the range of validity

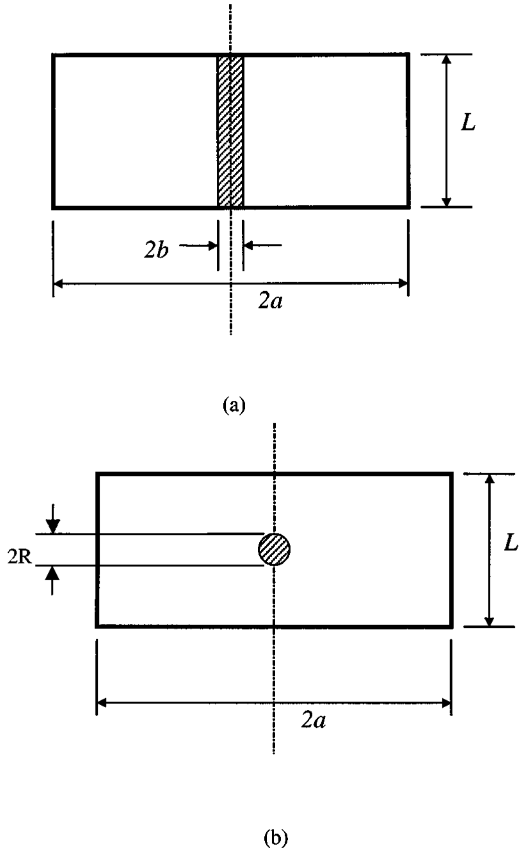

Fig. 1. Pill-box cavity resonator perturbed by: (a) a dielectric rod and (b) a dielectric sphere.

of the assumptions has not been checked. In the following sections, we examine this problem by comparing the approximate solutions with those obtained by direct application of (12).

III. Perturbation of a Pill-Box Cavity by a Dielectric Rod

Consider a pill-box cavity, excited in the TM_{100} mode, which is perturbed by a cylindrical dielectric rod placed along its axis, as shown in Fig. 1(a). The general solutions for the electric field inside and outside the rod are

$$E_1 = E_z = AJ_0\left(\sqrt{\varepsilon_r}k_1 r\right) \tag{20}$$

$$E_1 = E_z = BJ_0(k_1 r) + CY_0(k_1 r) \tag{21}$$

where A, B and C are constants, J_0 and Y_0 are the Bessel functions of the first and second kinds, and $k_1 = \omega_1/c$, where c is the velocity of light in vacuum. We can choose $A = 1$ without loss of generality. The constants B and C are determined by requiring that E_z and $\partial E_z/\partial r$ are continuous at the surface of the rod so that

$$B = \frac{Y_0(k_1 a) J_0\left(\sqrt{\varepsilon_r}k_1 b\right)}{J_0(k_1 b)Y_0(k_1 a) - J_0(k_1 a)Y_0(k_1 b)} \tag{22}$$

$$C = -B \cdot J_0(k_1 a)/Y_0(k_1 a). \tag{23}$$

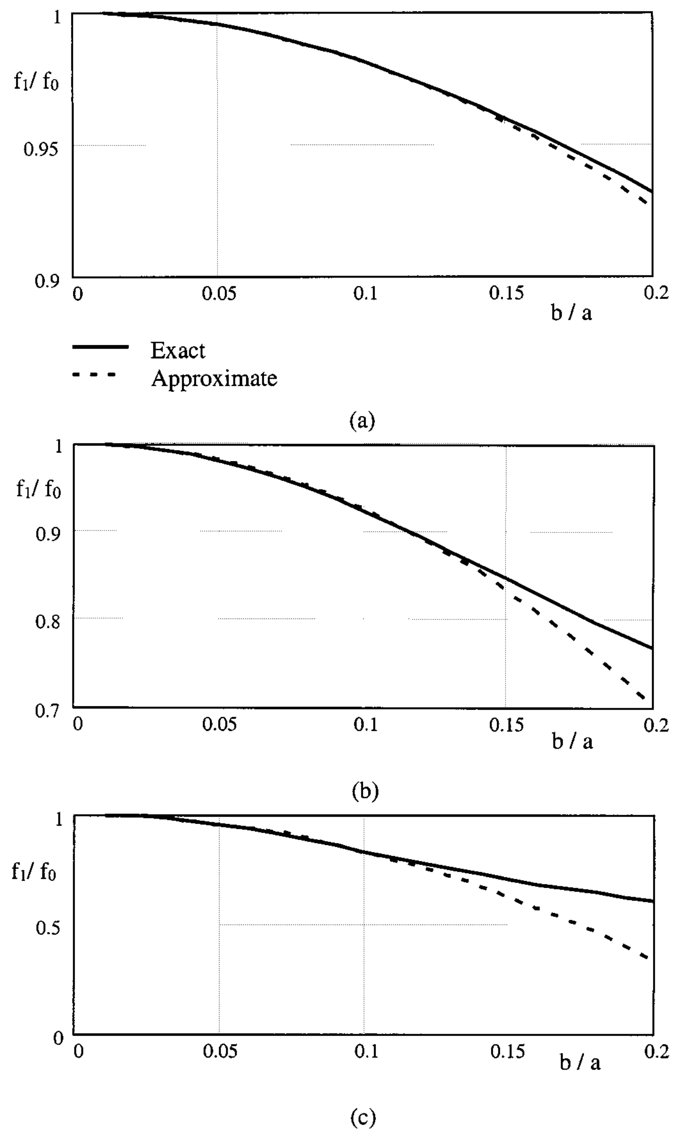

Fig. 2. Comparison of the resonant frequency of a pill-box cavity, perturbed by a dielectric rod, computed by exact and approximate methods. (a) $\varepsilon_r = 2$. (b) $\varepsilon_r = 5$. (c) $\varepsilon_r = 10$.

The requirement that E_z is zero at $r = a$ yields the determinantal equation

$$\sqrt{\varepsilon_r} \cdot \frac{J_1(\sqrt{\varepsilon_r}k_1 b)}{J_0(\sqrt{\varepsilon_r}k_1 b)} = \frac{J_1(k_1 b)Y_0(k_1 a) - J_0(k_1 a)Y_1(k_1 b)}{J_0(k_1 b)Y_0(k_1 a) - J_0(k_1 a)Y_0(k_1 b)}. \tag{24}$$

This equation can be solved numerically[1] to obtain k_1 and ω_1 for given values of a, b, and ε_r.

For the empty cavity, we note that

$$\omega_0 = k_0 c = 2.405\,(c/a). \tag{25}$$

Since the solutions scale directly with the dimensions, we can display the ratio ω_1/ω_0 against b/a for various values of ε_r, as shown in Fig. 2.

In the empty cavity, the electric field is given by

$$E_0 = E_z = J_0(k_0 r) \tag{26}$$

[1]The results presented in this paper were obtained using Mathcad8.

and the magnetic field by

$$H_0 = H_\theta = j\sqrt{\frac{\varepsilon_0}{\mu_0}}\, J_1(k_0 r). \tag{27}$$

In the perturbed cavity, the magnetic field is

$$H_1 = H_\theta = j\sqrt{\varepsilon_r}\sqrt{\frac{\varepsilon_0}{\mu_0}}\, J_1\big(\sqrt{\varepsilon_r} k_0 r\big) \tag{28}$$

inside the rod and

$$H_1 = H_\theta = j\sqrt{\frac{\varepsilon_0}{\mu_0}}\Big[BJ_1(k_1 r) + CY_1(k_1 r)\Big] \tag{29}$$

outside the rod.

When the fields defined by these equations are substituted into (12), the results are identical to those obtained from (24).

The stored energy in the empty cavity is given by

$$W_0 = \pi\varepsilon_0 L \int_0^a J_0(k_0 r)^2 r\, dr. \tag{30}$$

Substituting this expression into (17), and noting that $E_{00} = 1$, we obtain

$$\frac{\Delta\omega}{\omega_0} = 1.856\,(1-\varepsilon_r)(b/a)^2. \tag{31}$$

The frequency ratios computed from (31) for relative permittivities of two, five, and ten are compared with the exact results in Fig. 2. It is seen that there is good agreement between the two sets of results if $b/a \le 0.1$ and that the agreement deteriorates as b/a increases and as the relative permittivity increases. The accuracy is revealed more clearly in Fig. 3(a) and (b), which show the error in the approximate solutions for the ranges $0 < b/a < 0.1$ and $0 < b/a < 0.2$, respectively. If the normalized rod diameter is less than 0.1, the approximate solution is accurate to better than 1% for $\varepsilon_r \le 10$. If $\varepsilon_r = 2$, the difference between the exact and approximate formula is negligible. However, these results conceal possible sources of error, which make it unwise to assume that the same accuracies will apply to other shapes of the perturbing object.

Fig. 4 shows comparisons between the numerators and denominators of the exact and approximate expressions [see (12) and (17)]. From these, it is clear that the apparent accuracy of (17) is a consequence of the balancing of approximately equal errors in the numerator and denominator. These errors lie in the 1%–30% range for the cases investigated. Thus, the assumption that the second term in the denominator of (13) can be neglected is not as valid as has been generally supposed. The physical explanation of this result is that the electric field within the rod is overestimated by assumption 3 since the radial variation of the field within the rod has been neglected. The field outside the rod is reduced by the presence of the rod so that assumption 2 causes the denominator to be overestimated. It is fortuitous that the errors compensate each other in this case, but it is not safe to assume that a similar cancellation will occur in other cases. It is, therefore, possible that measurements made using perturbation methods may be in error by several percent.

One of the main uses of this theory is to determine the relative permittivities of samples of dielectric material in the form of rods. Since the method relies on the frequency perturbation caused by the rod, it is sensitive to quite small errors in the value of the perturbed frequency. This is illustrated in Fig. 5, in which values of the frequency perturbation obtained from the exact theory have been used to obtain the relative permittivity from (31). It can be seen that appreciable errors occur when the relative permittivity is calculated by the approximate method.

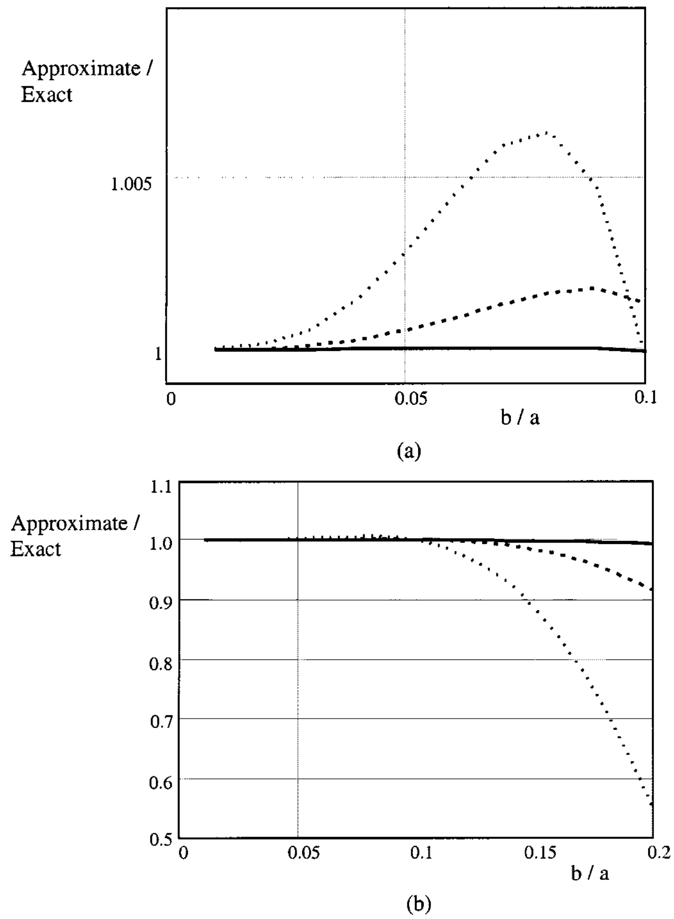

Fig. 3. Error in the resonant frequency of a pill-box cavity, perturbed by a dielectric rod, computed by the approximate method. (a) For $b/a \le 0.1$ and (b) $b/a \le 0.2$ for relative permittivities: $\varepsilon_r = 2$ (——), 5 (- - - - -), and 10 (. . .).

IV. Perturbation of a Pill-Box Cavity by a Dielectric Sphere

When a dielectric sphere is placed in a uniform electric field, the field within the sphere is given by (18) and the additional electric-field components outside the sphere produced by the polarization of the sphere are [7]

$$E_\rho = E_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) R^3 \left(\frac{2\cos\phi}{\rho^3}\right) \tag{32}$$

$$E_\phi = E_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) R^3 \left(\frac{\sin\phi}{\rho^3}\right) \tag{33}$$

in spherical polar coordinates.

We will assume that the dielectric sphere is placed on the axis of a pill-box cavity, as shown in Fig. 1(b). In order to be able to compute the frequency perturbation from (12), it is necessary to make two assumptions.

Assumption 5: The sphere is small enough for the field in which it is placed to be effectively constant. If we require the variation of the field to not be more than 1% over the space

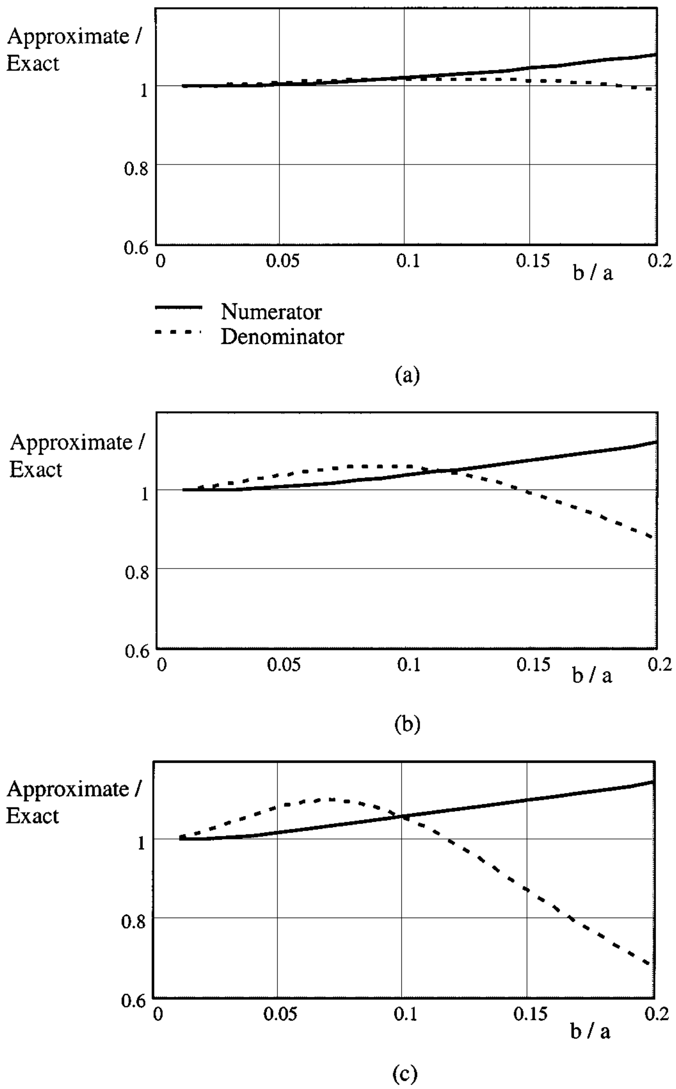

Fig. 4. Comparison between the numerators and denominators of the exact and approximate formulas for computing the resonant frequency of a pill-box cavity, perturbed by a dielectric rod. (a) $\varepsilon_r = 2$. (b) $\varepsilon_r = 5$. (c) $\varepsilon_r = 10$.

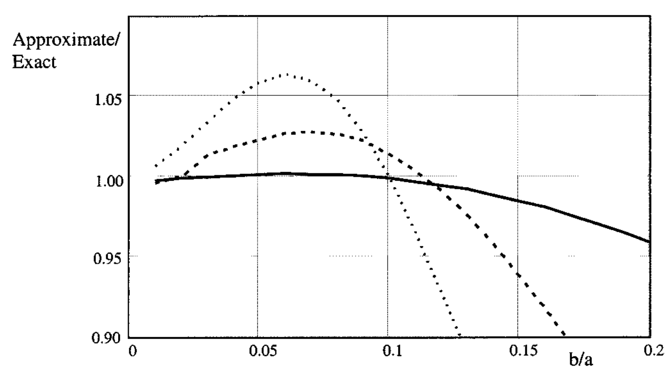

Fig. 5. Error in the calculation of the relative permittivity of a dielectric rod using the approximate formula for relative permittivities: $\varepsilon_r = 2$ (——), 5 (- - - - -), and 10 (. . .).

occupied by the sphere, then $k_0R = 0.2$ and, thus, $R/a \leq 0.083$. For a 5% field variation, $R/a \leq 0.19$.

Assumption 6: The sphere is small enough for the perturbation of the external field to be effectively zero on the boundary

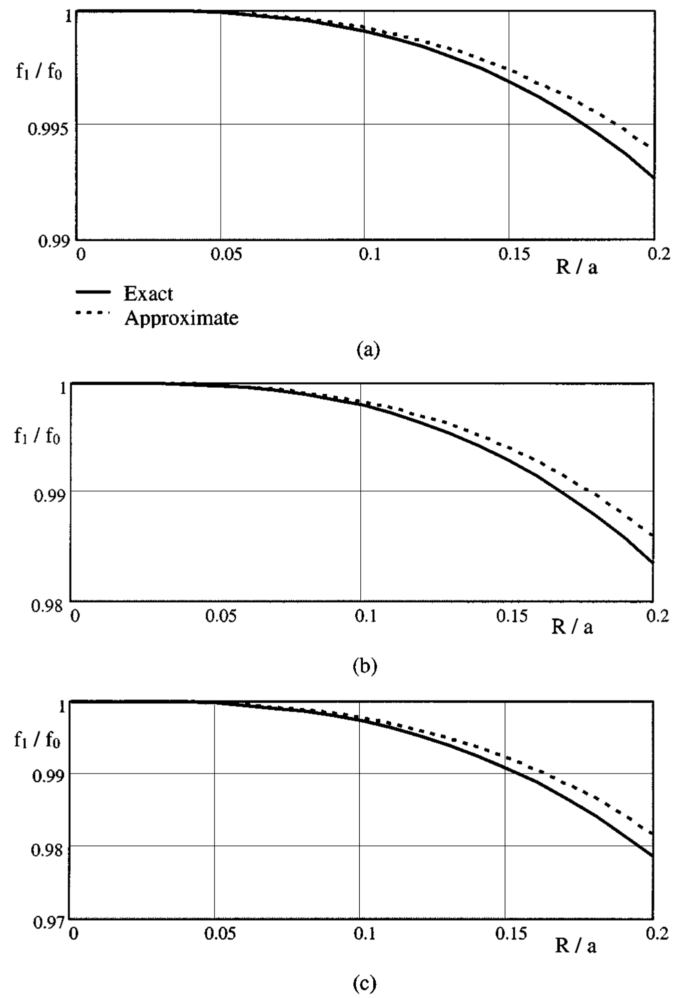

Fig. 6. Comparison of the resonant frequency of a pill-box cavity, perturbed by a dielectric sphere, computed by exact and approximate methods. (a) $\varepsilon_r = 2$. (b) $\varepsilon_r = 5$. (c) $\varepsilon_r = 10$.

of the cavity. If we set a limit of 1% on the perturbation, then R/a and $2R/L \leq 0.2$. Thus, for most cavities, the second condition will be satisfied whenever the first condition is true.

The field components outside the sphere are given, in cylindrical polar coordinates, by

$$E_{1z} = E_{0z} + E_\rho \cos\phi - E_\phi \sin\phi \tag{34}$$

$$H_{1\theta} = \frac{-j}{\mu_0 \omega_1} \frac{\partial E_{1z}}{\partial r} \tag{35}$$

where ω_1 is the as yet unknown perturbed frequency. The remaining field components are not required because their inner products with the unperturbed field components are zero. Within the sphere, for consistency, we must take $\boldsymbol{E}_0 = E_{00}$ and $\boldsymbol{H}_{1\theta} = 0$. Equation (12) can then be evaluated numerically to obtain values for the frequency perturbation, which are exact for small spheres. Fig. 6 shows how the ratio of the perturbed to the unperturbed frequency depends upon the radius and the relative permittivity of the sphere, as found from the approximate and exact calculations. Since we have used the same expression for the electric field inside the sphere for both the approximate and exact calculations, it follows that Fig. 6 shows the effect of neglecting the second term in the denominator of (13) in this case.

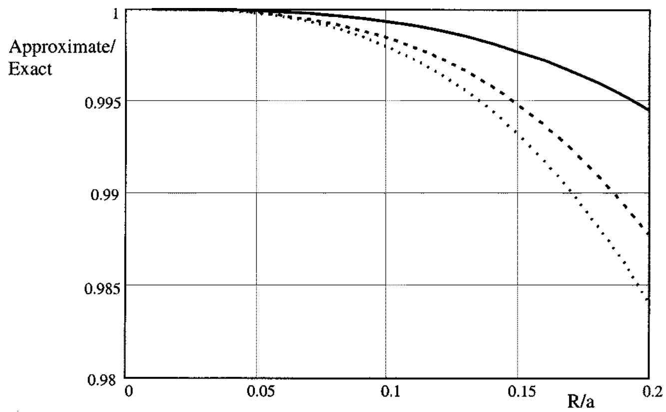

Fig. 7. Error in the magnitude of the electric field in a pill-box cavity, perturbed by a dielectric sphere, computed by the approximate method for relative permittivities. $\varepsilon_r = 2$ (——), 5 (- - - - -), and 10 (. . .).

The error introduced by this assumption is much less than in the case of perturbation by a rod because of the much smaller change in the fields outside the perturber.

Perturbation measurements using a dielectric sphere are commonly used to determine the electric-field distribution within a microwave structure. By substituting the frequency perturbation computed from (12) into (19), we can find the error in the determination of the field. The results of this calculation in Fig. 7 show that the error is less than 1% for typical sizes of sphere.

V. Conclusions

The results presented in this paper have shown that the assumptions made in the approximate theory of the perturbation of cavities by dielectric objects are not always valid. In particular, we have seen that the figures for the relative permittivity of dielectric rods may be in error by 5% for typical rod sizes. If the method is used to find the relative permittivity of rods having a uniform, but noncircular cross section, it is likely that similar accuracies will be obtained. When perturbation methods are used to characterize cavity resonators and other microwave structures, it is likely that the relative permittivity of the perturber will have been obtained by a perturbation measurement. In that case, the errors in measurement should be small provided that the same assumptions were made in interpreting both measurements and that the assumption that the perturber is located in a region of an uniform electric field is satisfied to a good approximation.

Acknowledgment

The results presented in this paper were obtained using Mathcad8.

References

[1] R. A. Waldron, "Perturbation theory of resonant cavities," *Proc. IEE*, vol. 107C, pp. 272–274, Sept. 1960.

[2] A. W. Kraszewski and S. O. Nelson, "Observations on resonant cavity perturbation by dielectric objects," *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp. 151–155, Jan. 1992.

[3] H. M. Altschuler, *Handbook of Microwave Measurements*, 3rd ed, M. Sucher and J. Fox, Eds. Brooklyn, NY: Polytech. Inst. Brooklyn, 1963, ch. Dielectric Constant.

[4] A. W. Horsley and A. Pearson, "Measurement of dispersion and interaction impedance characteristics of slow-wave structures by resonance methods," *IEEE Trans. Electron Devices*, vol. ED-13, pp. 962–969, Dec. 1966.

[5] D. Connolly, "Determination of the interaction impedance of coupled-cavity slow-wave structures," *IEEE Trans. Electron Devices*, vol. ED-23, pp. 491–493, May 1976.

[6] S. M. Hanna, G. B. Bowden, H. A. Hoag, R. J. Loewen, R. L. Miller, R. D. Ruth, and J. W. Wang, "Microwave cold- testing techniques for the NLC," in *Proc. European Particle Accelerator Conf.*, 1996, pp. 2056–2058.

[7] B. I. Bleaney and B. Bleaney, *Electricity and Magnetism*. Oxford, U.K.: Oxford Univ. Press, 1957.

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